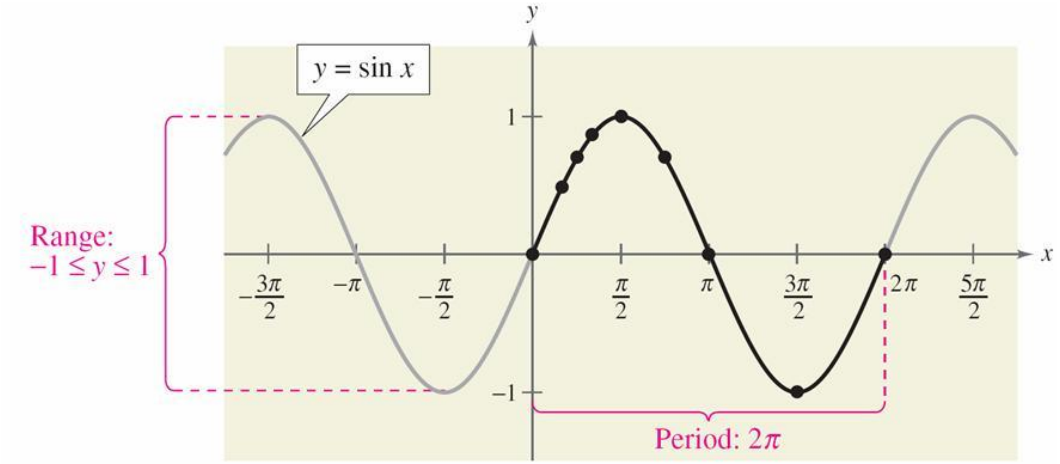
4.5 GRAPHS OF SINE AND COSINE FUNCTIONS

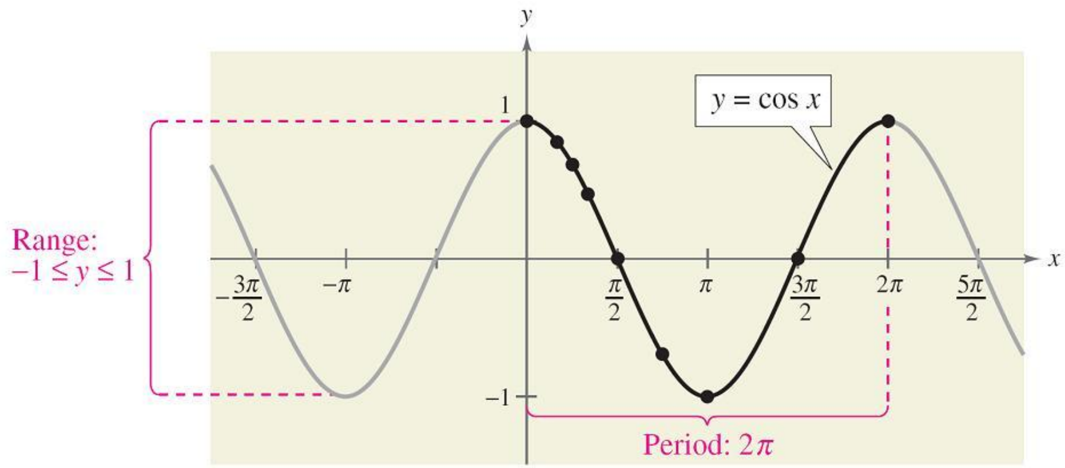
What You Should Learn

- Sketch the graphs of basic sine and cosine functions.
- Use amplitude and period to help sketch the graphs of sine and cosine functions.
- Sketch translations of the graphs of sine and cosine functions.
- Use sine and cosine functions to model real-life data.

Basic Sine and Cosine Curves

Basic Sine and Cosine Curves





The black portion of the graph represents one period of the function and is called **one cycle** of the sine curve.

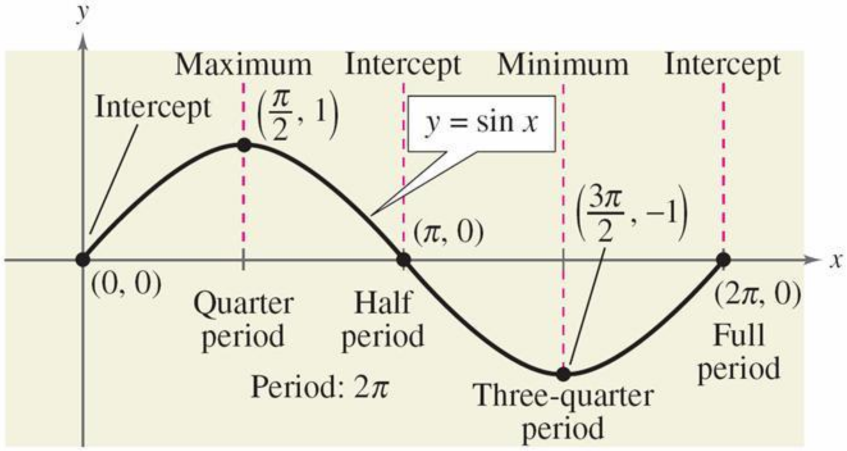
The domain of the sine and cosine functions is the set of all real numbers.

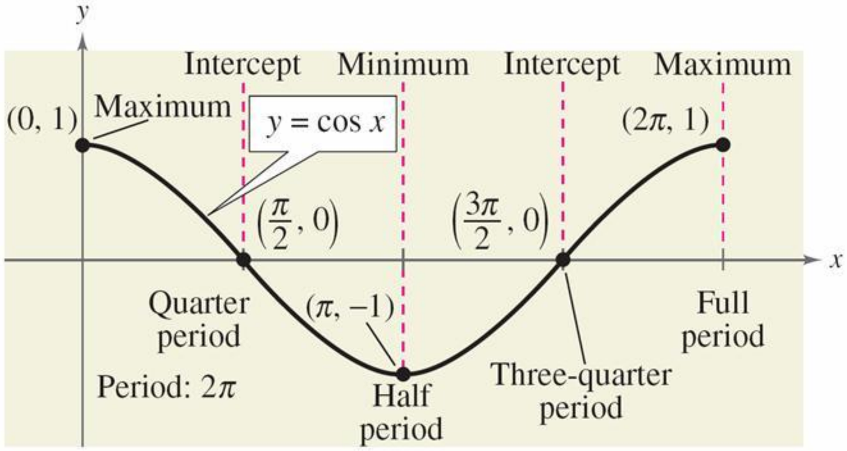
The range of each function is the interval [–1, 1].

Each function has a period of 2π.

Basic Sine and Cosine Curves

Five **key points** in one period of each graph: the *intercepts*, *maximum points*, and *minimum points*





Example 1 – *Using Key Points to Sketch a Sine Curve*

Sketch the graph of $y = 2 \sin x$ on the interval $[-\pi, 4\pi]$.

Solution:

Note that

$$y = 2 \sin x = 2(\sin x)$$

indicates that the y-values for the key points will have twice the magnitude of those on the graph of $y = \sin x$.

Divide the period 2π into four equal parts to get the key points for $y = 2 \sin x$.

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$(0, 0)$,	$\left(\frac{\pi}{2}, 2\right)$,	$(\pi, 0)$,	$\left(\frac{3\pi}{2}, -2\right)$,	and	$(2\pi, 0)$

Example 1 – *Solution*

cont'd

By connecting these key points with a smooth curve and extending the curve in both directions over the interval $[-\pi, 4\pi]$, you obtain the graph shown in Figure 4.50.

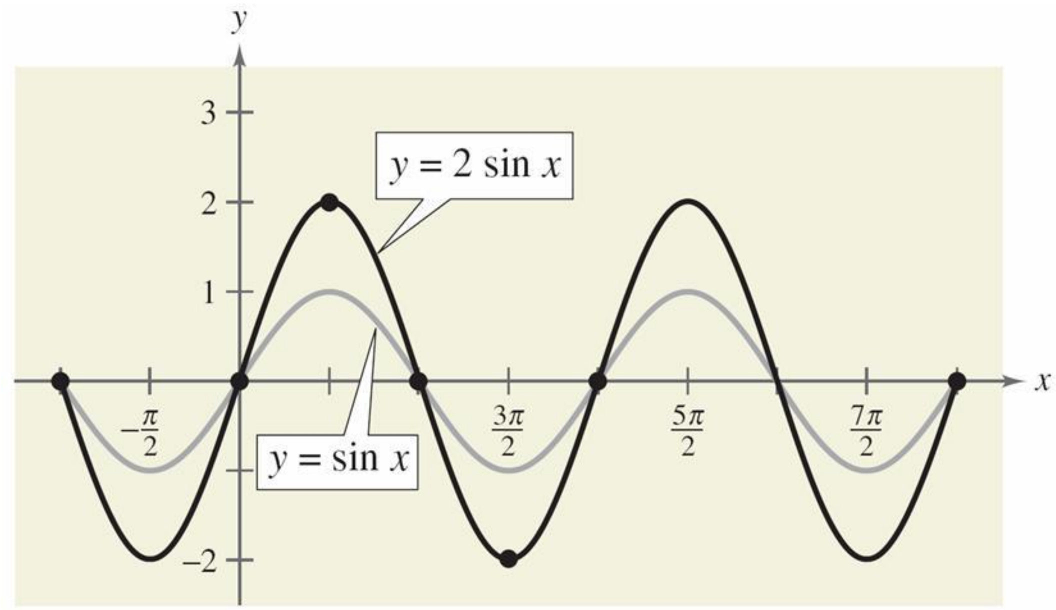


Figure 4.50

Amplitude and Period

Amplitude and Period

$$y = d + a \sin(bx - c)$$

and

$$y = d + a \cos(bx - c).$$

If $|a| > 1$, the basic sine curve is stretched,

If $|a| < 1$, the basic sine curve is shrunk.

The result is that the graph of $y = a \sin x$ ranges between $-a$ and a instead of between -1 and 1.

The range of the function $y = a \sin x$ for $a > 0$ is $-a \le y \le a$.

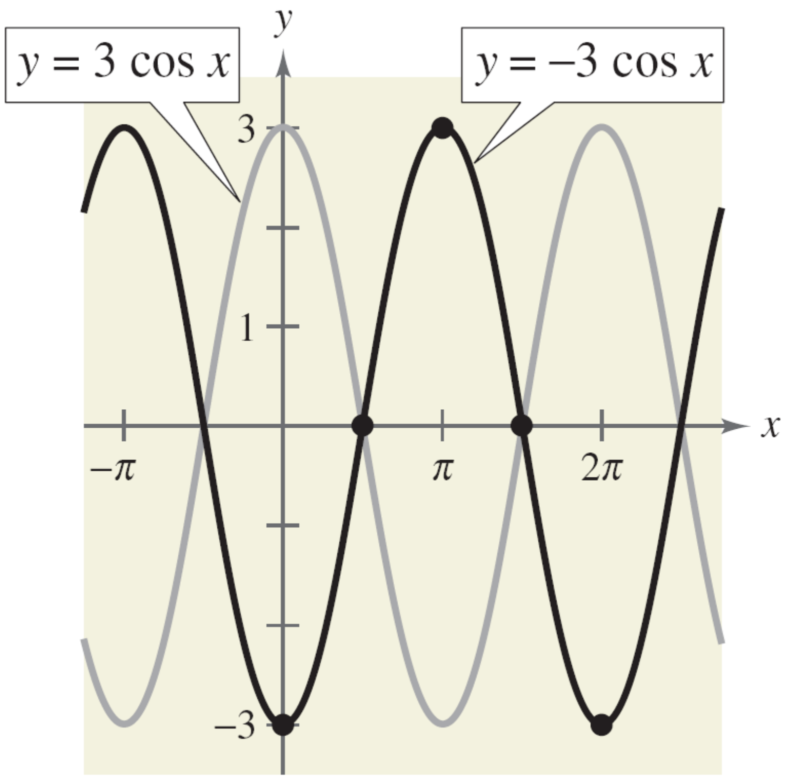
Amplitude and Period

Definition of Amplitude of Sine and Cosine Curves

The **amplitude** of $y = a \sin x$ and $y = a \cos x$ represents half the distance between the maximum and minimum values of the function and is given by

$$\text{Amplitude} = |a|.$$

Amplitude and Period



Amplitude and Period

Period of Sine and Cosine Functions

Let b be a positive real number. The **period** of $y = a \sin bx$ and $y = a \cos bx$ is given by

$$\text{Period} = \frac{2\pi}{b}.$$

If $b > 1$, the period of $y = a \sin bx$ is less than 2π and represents a *horizontal shrinking* of the graph of $y = a \sin x$.

If b is negative, the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ are used to rewrite the function.

Example 3 – *Scaling: Horizontal Stretching*

Sketch the graph of $y = \sin \frac{x}{2}$.

Solution:

The amplitude is 1. Moreover, because $b = \frac{1}{2}$, the period is

$$\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}}$$

Substitute for *b*.

$$= 4\pi.$$

Example 3 – *Solution*

cont'd

Now, divide the period-interval $[0, 4\pi]$ into four equal parts with the values π, 2π, and 3π to obtain the key points on the graph.

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$(0, 0)$,	$(\pi, 1)$,	$(2\pi, 0)$,	$(3\pi, -1)$,	and	$(4\pi, 0)$

The graph is shown in Figure 4.53.

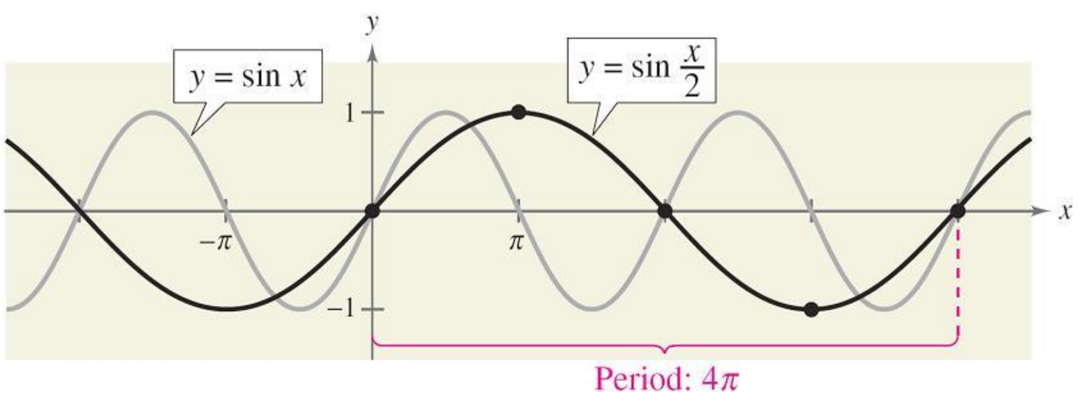


Figure 4.53

Translations of Sine and Cosine Curves

Translations of Sine and Cosine Curves

The constant *c* in the general equations

$$y = a \sin(bx - c) \qquad \text{and} \qquad y = a \cos(bx - c)$$

creates a *horizontal translation* (shift) of the basic sine and cosine curves.

$$\overbrace{\frac{c}{b}}^{\text{Left endpoint}} \le x \le \overbrace{\frac{c}{b} + \underbrace{\frac{2\pi}{b}}_{\text{Period}}}^{\text{Right endpoint}}.$$

The number *c*/*b* is the **phase shift.**

Translations of Sine and Cosine Curves

Graphs of Sine and Cosine Functions

The graphs of $y = a\sin(bx - c)$ and $y = a\cos(bx - c)$ have the following characteristics. (Assume $b > 0$.)

$$\text{Amplitude} = |a| \qquad \text{Period} = \frac{2\pi}{b}$$

The left and right endpoints of a one-cycle interval can be determined by solving the equations $bx - c = 0$ and $bx - c = 2\pi$.

Example 5 – *Horizontal Translation*

Sketch the graph of

$$y = –3 \cos(2\pi x + 4\pi).$$

Solution:

The amplitude is 3 and the period is $2\pi / 2\pi = 1$.

$$2\pi x + 4\pi = 0$$

$$2\pi x = –4\pi$$

$$x = –2$$

And $$2\pi x + 4\pi = 2\pi$$

$$2\pi x = –2\pi$$

$$x = –1$$

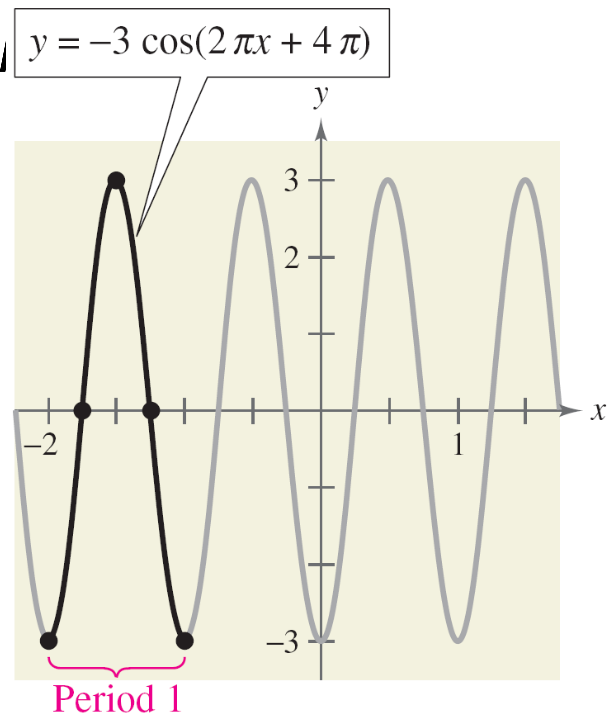
Example 5 – *Solution*

The interval [–2, –1] corresponds to one cycle of the graph.

Dividing this interval into four equal parts produces the key points

Minimum *Intercept* *Maxi* *inimum*

$(-2, -3)$, $\left(-\frac{7}{4}, 0\right)$, $\left(-\frac{3}{2},\right.$ $-1, -3)$.



Translations of Sine and Cosine Curves

The final type of transformation is the *vertical translation* caused by the constant d in the equations

$$y = d + a\sin(bx - c)$$

and

$$y = d + a\cos(bx - c).$$

The shift is d units upward for $d > 0$ and d units downward for $d < 0$.

The graph oscillates about the horizontal line $y = d$ instead of about the x-axis.

Translations of Sine and Cosine Curves

aptitude

Phase shift: c/b

$$y = a\sin(bx - c) + d$$

Period: $2\pi/b$

Vertical shift

Mathematical Modeling

Mathematical Modeling

Sine and cosine functions can be used to model many real-life situations, including electric currents, musical tones, radio waves, tides, and weather patterns.

Example 7 – *Finding a Trigonometric Model*

Throughout the day, the depth of water at the end of a dock in Bar Harbor, Maine varies with the tides. The table shows the depths (in feet) at various times during the morning. (Source: Nautical Software, Inc.)

Time, t	Depth, y
Midnight	3.4
2 A.M.	8.7
4 A.M.	11.3
6 A.M.	9.1
8 A.M.	3.8
10 A.M.	0.1
Noon	1.2

Example 7 – *Finding a Trigonometric Model* cont'd

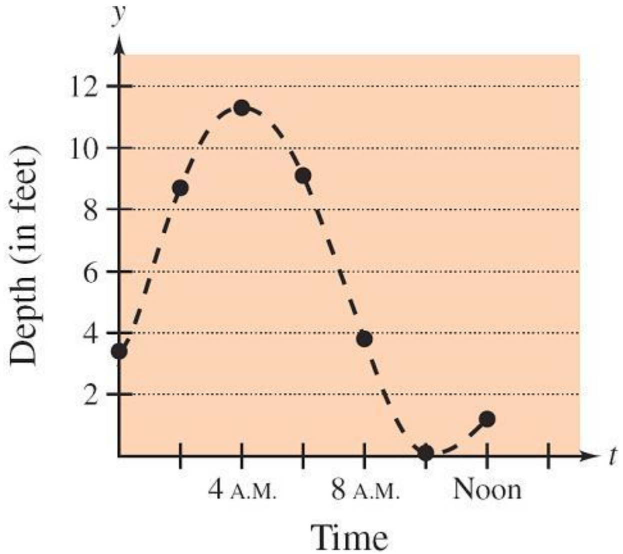
a. Use a trigonometric function to model the data.

b. Find the depths at 9 A.M. and 3 P.M.

c. A boat needs at least 10 feet of water to moor at the dock. During what times in the afternoon can it safely dock?

Example 7(a) – *Solution*

Begin by graphing the data, as shown in Figure 4.57.



Changing Tides

Figure 4.57

You can use either a sine or a cosine model. Suppose you use a cosine model of the form

$$y = a \cos(bt - c) + d.$$

Example 7(a) – *Solution*

cont'd

The difference between the maximum height and the minimum height of the graph is twice the amplitude of the function. So, the amplitude is

$$a = \frac{1}{2}[(\text{maximum depth}) - (\text{minimum depth})]$$

$$= \frac{1}{2}(11.3 - 0.1)$$

$$= 5.6.$$

The cosine function completes one half of a cycle between the times at which the maximum and minimum depths occur. So, the period is

$$p = 2[(\text{time of min. depth}) - (\text{time of max. depth})]$$

Example 7(a) – *Solution*

cont'd

$= 2(10 - 4)$

$= 12$

which implies that

$b = 2\pi/p$

$\approx 0.524.$

Because high tide occurs 4 hours after midnight, consider the left endpoint to be $c/b = 4$, so $c \approx 2.094$.

Example 7(a) – *Solution*

cont'd

Moreover, because the average depth is
$\frac{1}{2}(11.3 + 0.1) = 5.7$, it follows that $d = 5.7$.

So, you can model the depth with the function given by

$$y = 5.6 \cos(0.524t - 2.094) + 5.7.$$

Example 7(b) – *Solution*

cont'd

The depths at 9 A.M. and 3 P.M. are as follows.

$$y = 5.6 \cos(0.524 \cdot 9 - 2.094) + 5.7$$

$$\approx 0.84 \text{ foot} \qquad \text{9 A.M.}$$

$$y = 5.6 \cos(0.524 \cdot 15 - 2.094) + 5.7$$

$$\approx 10.57 \text{ foot} \qquad \text{3 P.M.}$$

Example 7(c) – *Solution*

cont'd

To find out when the depth *y* is at least 10 feet, you can graph the model with the line *y* = 10 using a graphing utility, as shown in Figure 4.58.

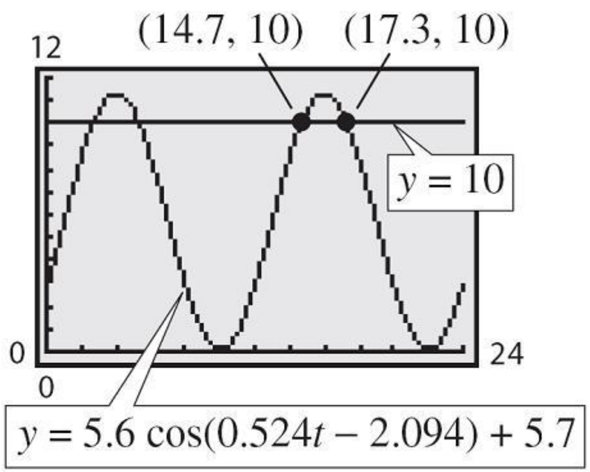


Figure 4.58

Using the *intersect* feature, you can determine that the depth is at least 10 feet between 2:42 P.M. ($t \approx 14.7$) and 5:18 P.M. ($t \approx 17.3$).